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Statistical equilibrium states for two-dimensional flows

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We explain the emergence of organized structures in two-dimensional turbulent flows by a theory of equilibrium statistical mechanics. This theory takes into account all the known constants of the motion for the Euler equations. The microscopic states are all the possible vorticity fields, while a macroscopic state is defined as a probability distribution of vorticity at each point of the domain, which describes in a statistical sense the fine-scale vorticity fluctuations. The organized structure appears as a state of maximal entropy, with the constraints of all the constants of the motion. The vorticity field obtained as the local average of this optimal macrostate is a steady solution of the Euler equation. The variational problem provides an explicit relationship between stream function and vorticity, which characterizes this steady state. Inertial structures in geophysical fluid dynamics can be predicted, using a generalization of the theory to potential vorticity.

1. Introduction

The appearance of coherent structures is one of the most striking features of two-dimensional turbulence. While there is an obvious tendency in ordinary fluid turbulence for the system to try to increase its disorder, at the same time there are circumstances in which a sort of 'macroscopic' order seems to emerge from what appears to be 'microscopic' disorder. (The term microscopic does not refer here to any molecular motion but to a fine-scale turbulent field in a continuous medium.) As we are interested in situations where the dynamics is dominated by the inertial terms (high Reynolds number), we make the reasonable assumption that the fluid is incompressible and perfect, and so its dynamics is governed by the classical incompressible Euler system. Then the straight question is: how can this behaviour of the fluid be explained or predicted from this set of equations? The observation of the merging of two like-sign vortices, experimentally (Hopfinger 1989; Capéran 1989; Sommeria, Meyers & Swinney 1988) or in numerical simulations (Overman & Zabusky 1982), shows that the final 'macroscopic' state does not depend on the very variable nature of the intermediate 'microscopic' states due to the complicated deformation of the vortices by mutual straining. This suggests that an explanation of the phenomenon must be of a statistical nature. Of course this conclusion is not new, and there have been several attempts to build up some statistical hydrodynamics, beginning with the pioneering work of Onsager (1949).

We can classify all attempts to apply the methods of statistical mechanics to fluid dynamics into two categories. In the first one we find several works which have continued Onsager's approach. The idea was to approximate the continuous Euler

system by a great (but finite) number of point vortices. This leads to a finite-dimensional Hamiltonian system, to which the methods of statistical mechanics can be applied; see for example Novikov (1976), Poitin & Lundgren (1976). Particularly interesting presentations and discussions of this approach can be found in Montgomery (1985) and Saffman & Baker (1979). Though very enlightening, this approach reveals a severe difficulty. There are many different ways to approximate a continuous vorticity by a cloud of point vortices. And different approximations can lead to very different statistical equilibrium states. So, the thermodynamical equilibrium state that we can associate to a continuous vorticity depends dramatically on arbitrary choices (this difficulty was underlined by Onsager).

There is another way to approximate the two-dimensional Euler system: we decompose the vorticity into a Fourier series and truncate the description to a finite number of Fourier coefficients. One can prove a Liouville theorem for the truncated system (in the phase space of Fourier coefficients, the volume element is conserved). This suggests again that the methods of statistical mechanics be employed (Lee 1952; Kraichnan 1975; Kraichnan & Montgomery 1980). It happens that after the truncation, there remain only two constants of the motion: the energy, and the mean-square vorticity or enstrophy. Then the Gibbs canonical ensemble, corresponding to these two constraints is easily obtained. Here also there is a serious obstacle. When we consider the truncated system, instead of the full Euler system, we lose the information given by all the integral functionals of the vorticity (except enstrophy), which are constants of the motion for the full system (this is due to the law of vorticity conservation along the trajectories of the fluid particles). As a consequence, the significance of the equilibrium states of the truncated system for the full one is far from obvious. If the system were ergodic in some sense (this is the underlying hypothesis of any statistical mechanics approach) it can only be on the 'submanifold' of the phase space defined by all the constants of the motion fixed at their initial value. To overcome this difficulty, one can try to construct Gibbs states for the full Euler system by a limit process when the number of Fourier coefficients goes to infinity. This was done by Boldrighini & Frigio (1980) in a very interesting work. These authors succeeded in constructing a family of Gibbs states for the infinite-dimensional system, associated with the law of vorticity conservation along the trajectories of the fluid particles. Unfortunately, these probability measures are supported by very 'large' functional spaces of generalized functions; so that not only are the mean energy and enstrophy of these states infinite but the phase space of bounded measurable vorticity functions, on which the Eulerian flow can be defined, is of null measure. So, it is only at a formal level that this makes sense.

The main conclusion that can be drawn from this short overview is that, although the finite-dimensional approximations of Euler equations can provide a good representation of the flow during a finite time, the information that the thermodynamics (or long-time dynamics) of such systems gives on the behaviour of the full system is highly questionable.

To introduce our approach, let us briefly recall the standard ingredients of classical statistical mechanics. We start with a Hamiltonian system which gives the dynamics of a great number of particles and which is the microscopic level of description of the system. Then, at a macroscopic level, we consider some relevant means, which we call macroscopic observables. To these two levels of description we associate an entropy functional using Boltzmann's formula $S = k \log W$, where W is the volume occupied in the phase space (endowed with the invariant Liouville measure) by the set of all the microstates giving the same macrostate. Maximizing the entropy functional then

gives the equilibrium states. Setting apart the ergodic aspect of the problem, the justification of the method comes from a concentration property: an overwhelming majority of microstates correspond to the maximum entropy macrostate; so that it has a high probability of being observed. This concentration property (which can be proved in the general framework of large-deviation theory) is essential as it justifies the use of Boltzmann's entropy.

Now, when we work with the Euler system, we deal with the infinite-dimensional phase space of bounded vorticity functions on which the Eulerian flow is defined and which gives the microscopic level of description. Although the Euler equation has an Hamiltonian structure (Arnol'd 1966; Olver 1986), no Liouville theorem is known, so we use a more general point of view of statistical mechanics. Such an approach, inspired by Jaynes (1985), provides a method of finding the 'most probable' probability law for a random process, among all the possible laws compatible with the known information on the system. Remarkable success has been obtained with this approach in the very different fields of image analysis and data processing, and the power of the method is justified on a firm mathematical basis by a theorem of concentration (Robert 1989). We give a macroscopic description of the system by means of Young measures: at each point we have a probability distribution of vorticity which gives, in some statistical sense, a local description of the small-scale oscillations of the microscopic vorticity functions. Then, although we no longer have a Liouville measure at our disposal, we define an entropy functional on the set of macrostates. It is the Kullback entropy of probability theory, which appears as a straightforward generalization of Boltzmann's mixing entropy. Maximizing the entropy gives the equilibrium states. The method is justified as we can prove that the maximum-entropy state satisfies a natural concentration property, which is conserved by the Eulerian flow, in the phase space.

Besides the fact that we work with the full Euler system and take into account all the known constants of the motion, our approach has the following advantages. First, we can provide mathematical proofs of the essential properties of concentration and invariance. Furthermore, it gives clear formulae that permit quantitative comparison to experiments and numerical simulations (Sommeria, Staquet & Robert 1991). After brief summaries of the two-dimensional Euler equations in §2, the theory is explained in §3 using simple physical arguments. The resulting equations and their consequences are discussed in §§5–7. Many doubtful principles of maximum entropy have been proposed, using vague physical arguments. Therefore it is important to precisely state the mathematical justification of the present theory, and this is done in §4, referring to the papers by Robert (1989, 1990, 1991) for the proofs of the theorems. Nevertheless the physical meaning and the consequences of the theory can be understood without this §4, and the reader can proceed directly to §5.2.

2. The two-dimensional incompressible Euler equation

2.1. *The equations*

Throughout this paper, we shall work with the solutions of the incompressible Euler system in an open bounded regular and simply connected domain Ω of the plane. Let $\boldsymbol{u}(t,\boldsymbol{x})$ be the velocity field of the fluid, we find it convenient to introduce the scalar vorticity $\omega(t,\boldsymbol{x}) = \nabla\times\boldsymbol{u}(t,\boldsymbol{x})$, and to write the system in the velocity–vorticity formulation:

$$\left.\begin{aligned}\omega_t+\nabla\cdot(\omega\boldsymbol{u}) &= 0,\\ \omega(0,\boldsymbol{x}) &= \omega_0(\boldsymbol{x}),\\ \nabla\times\boldsymbol{u} &= \omega,\end{aligned}\right\} \tag{1a}$$

$$\left.\begin{aligned} &\nabla\cdot\boldsymbol{u} = 0, \\ &\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega \quad (\boldsymbol{n} \text{ normal to } \partial\Omega. \end{aligned}\right\} \tag{1b}$$

Thus the Euler system appears as a transport equation (1*a*) coupled with the elliptic system (1*b*).

The system (1*b*) is classically solved by introducing the stream function ψ:

$$-\nabla^2\psi = \omega,$$
$$\psi = 0 \quad \text{on} \quad \partial\Omega.$$

Then $\boldsymbol{u} = \nabla \times (\psi\boldsymbol{z})$ where $\boldsymbol{z}$ is the normal unit vector) gives the solution of (1*b*). In the same way, for a given velocity field $\boldsymbol{u}(t, \boldsymbol{x})$, we can solve (1*a*) by introducing the Lagrangian flow $\phi_t(\boldsymbol{x})$, defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(\boldsymbol{x}) = \boldsymbol{u}(t, \phi_t(\boldsymbol{x})) \quad \text{for all } \boldsymbol{x} \text{ in } \Omega,$$

$$\phi_0(\boldsymbol{x}) = \boldsymbol{x}.$$

One can easily check that the map ϕ_t is area-preserving ($\boldsymbol{u}$ is incompressible) and that the vorticity ω is transported by ϕ_t:

$$\omega(t, \phi_t(\boldsymbol{x})) = \omega_0(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \text{ in } \Omega.$$

It is well known that if ω_0 is any regular function on $\bar{\Omega}$, (1*a*, *b*) has a unique classical solution for all time (Youdovitch 1963; Kato 1967; Bardos 1972). Using the notion of weak solution (in the sense of generalized functions) one can extend this property to the case where ω_0 is in the space $L^\infty(\Omega)$ of all bounded measurable functions on Ω (Cottet 1987). Thus the space $L^\infty(\Omega)$ is a convenient choice for our phase space, and we denote $\boldsymbol{\Phi}_t : L^\infty(\Omega) \to L^\infty(\Omega)$ the Eulerian flow. $\boldsymbol{\Phi}_t$ is related to the above Lagrangian flow ϕ_t by the relation:

$$\boldsymbol{\Phi}_t\,\omega_0(\boldsymbol{x}) = \omega_0(\phi_t^{-1}(\boldsymbol{x})).$$

2.2. *Stationary solutions*

In the case of stationary solutions, (1*a*, *b*) reduces to

$$\nabla\cdot(\omega\nabla \times (\psi\boldsymbol{z})) = 0.$$

This is satisfied for example if $\omega = f(\psi)$, where f is any continuous function on $\mathbb{R}$, or if Ω is a ball centred at 0 and ω is rotation invariant.

2.3. *Constants of the motion*

One can easily check that the following functionals are conserved by the Eulerian flow:

the energy $$E(\omega) = \frac{1}{2}\int_\Omega u^2\,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\int_\Omega \psi\omega\,\mathrm{d}\boldsymbol{x};$$

the integrals $$I_f(\omega) = \int_\Omega f(\omega(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}, \text{ for any continuous function } f;$$

if Ω is the ball $B(0, R)$, we must consider also the angular momentum with respect to 0:

$$\int_\Omega \boldsymbol{x} \wedge \boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\int_\Omega (R^2 - \boldsymbol{x}^2)\,\omega(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x};$$

in the case of periodic boundary conditions, we must take into account the two components of the linear momentum $\int_\Omega \boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ instead of the angular momentum; if Ω is not simply connected, there are new constants of the motion given by the circulation around the obstacles.

It can be shown (Serre 1984) that there are no invariant functionals of the form

$$\int_\Omega F(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}$$

other than those we already know.

2.4. *Expressing the local conservation of vorticity*

It will be convenient to express the local conservation of vorticity in a more abstract way. For any (measurable) bounded vorticity function ω, we define the measure π_ω on $\mathbb{R}$ by

$$\langle \pi_\omega, f \rangle = \int_\Omega f(\omega(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}.$$

π_ω is called the distribution measure of the function ω. It follows from the conservation of $I_f(\omega)$ for any continuous function f that the measure π_ω is conserved by the Eulerian flow: $\pi_{\Phi_t\omega_0} = \pi_{\omega_0}$. We shall say that a vorticity field ω' is a reordering of ω if it satisfies $\pi_\omega = \pi_{\omega'}$. In other words, the vorticity at time t, $\Phi_t\omega_0$, is a reordering of the initial vorticity ω_0. If ω_0 is made of n patches Ω_i of value a_i and area $|\Omega_i|$, then $I_f(\omega_0) = \sum_i |\Omega_i| f(a_i)$. Therefore π_ω is a sum of Dirac masses δ_{a_i}: $\pi_\omega = \sum_i |\Omega_i|\,\delta_{a_i}$. So we see that giving the measure π_ω is equivalent to giving all the vorticity levels and their area. The conservation of π_ω by the flow represents the conservation of all these quantities.

3. A macroscopic description of intricate vorticity fields

The actual flow evolution from the initial vorticity field ω_0 is tremendously complex and difficult to predict. We only know that the constants of the motion must be conserved. Therefore, instead of seeking a deterministic prediction of the flow, we consider all the possible vorticity fields $\xi(x)$ consistent with the conserved quantities. A central result of this paper is that an overwhelming majority of such microstates $\xi(x)$ is in a neighbourhood of a well-defined macrostate. Therefore, if some ergodic property were satisfied, the actual flow evolution would be very likely to stay near this macrostate. Of course, we must precisely define a macrostate and the notion of proximity with microstates. Since we are in an infinite-dimensional function space, this is technically complex, and the details are given in §4. However, a simplified heuristical derivation by combinatorial arguments is presented below.

In this section, we restrict the analysis to an initial vorticity function made of n patches Ω_i of value a_i. Of course any initial vorticity field can be approximated by such a function with large n. As time goes on, the function $\Phi_t\omega_0$ still takes the same values a_i, but the vortex patches generally become more and more intricate. Numerical computations by contour dynamics (Zabuski, Hughes & Roberts 1979; Dritschel 1989) give a good intuition of this highly complex process (but any numerical method is limited by the requirement of higher and higher spatial resolution as time goes on). Then, it will be convenient to introduce the new macroscopic variables $e_i(\boldsymbol{x})$, $i = 1 \ldots n$, which give, at each point $\boldsymbol{x}$, the probability of finding the value a_i at point $\boldsymbol{x}$. Such a probability field $e(\boldsymbol{x}) = (e_1(\boldsymbol{x}), \ldots e_n(\boldsymbol{x}))$ will be called a macrostate.

Since most of the microscopic fields have very fine vorticity structures, we can describe them in terms of a local fraction of area for the different vorticity levels in a small neighbourhood of each point x. By assimilating this local fraction into a probability distribution, we can then associate a macrostate with each microstate. The constants of the motion bring constraints to the macrostates, as discussed now. The angular momentum is indeed an integral of the form

$$\int_\Omega f(x)\,\xi(x)\,\mathrm{d}x, \quad \text{where} \quad f(x) = \tfrac{1}{2}(R^2 - x^2)$$

is a smooth function of x. Therefore, it is justifiable to replace the rapidly oscillating vorticity $\xi(x)$ by its local average $\bar{\omega}(x) = \sum_i a_i e_i(x)$ in the integral. Since the angular momentum must remain equal to its value in the initial condition, a constraint on the macrostate is thus obtained.

Similarly, it is classically known that the stream function of ξ is continuous so that

$$E(\xi) = \frac{1}{2}\int_\Omega \bar{\omega}(x)\,\psi(x)\,\mathrm{d}x.$$

We define the macroscopic stream function Ψ by $-\nabla^2\Psi = \bar{\omega}$. (Notice that the macrostate has fluctuations in vorticity, but as Ψ is obtained by an integration of vorticity, there are no fluctuations in stream function.) Then an integration by part gives

$$E(\xi) = \frac{1}{2}\int_\Omega \xi\Psi\,\mathrm{d}x,$$

and the same argument as above yields

$$E(\xi) = \frac{1}{2}\int_\Omega \bar{\omega}\Psi\,\mathrm{d}x = E(\bar{\omega}).$$

In other words, the fine-scale vorticity fluctuations do not contribute to the energy.

As we have seen, the functionals of the form $I_f(\omega)$ are expressed as $I_f(\omega) = \sum_i |\Omega_i| f(a_i)$, and the conservation of all these functions is equivalent to the conservation of the area of each vorticity patch. Since $e_i(x)$ represents the local proportion of this area with respect to the local area element $\mathrm{d}x$, the conservation of the area of each patch gives

$$\int_\Omega e_i(x)\,\mathrm{d}x = |\Omega_i| \quad \text{for all } i.$$

Now, we want to introduce the entropy of a macrostate $e(x) = (e_1(x), \ldots, e_n(x))$. For that purpose, let us consider a small neighbourhood of the point x and split it into N non-intersecting pieces of equal area. Then, we restrict consideration to the subset of microstates which take a constant value on each piece. With each such microstate we associate the probability distribution $(e_1, \ldots, e_n)$ given by $e_i = N_i/N$, $i = 1, \ldots, n$, where N_i is the number of pieces where the value a_i is reached. With each probability distribution $(e_1, \ldots, e_n)$, we associate the number of ways by which it can be obtained or multiplicity factor:

$$W(e_1, \ldots, e_n) = \frac{N!}{(e_1 N)! \ldots (e_n N)!}.$$

Then we see, by an elementary analysis, that the normalized expression $(1/N)\log W(e_1, \ldots, e_n)$ gives at the limit, when N becomes infinite, the Shannon classical entropy:

$$\frac{1}{N}\log W \underset{N\to\infty}{\to} -\sum_{i=1}^{n} e_i \log e_i.$$

Now, if we consider the whole domain Ω, the total entropy is clearly the sum of the contributions of each area element $\mathrm{d}x$, so the entropy of the macrostate $e(x)$ is

$$S(e) = -\int_\Omega \sum_i e_i(x) \log e_i(x)\, \mathrm{d}x.$$

This expression is well known as Boltzmann's mixing entropy. Then, if we assume that all the microscopic states are equiprobable, the maximum-entropy macrostate satisfying the constraints given by the constants of the motion is the most probable macrostate; that is, it corresponds to the greatest number of microstates satisfying the given constraints. Furthermore, it is clear from large-deviation theory that the fluctuations would go to zero as the number of parcels N goes to infinity, so that with a probability very close to 1 the system will actually be very close to the most probable state. Notice however that, while this idea corresponds to a 'popular wisdom', it is not easy to prove in our particular case, because of the nonlinear constraint given by the energy. The precise formulation is presented in §4. Finer and finer scales are reached as time goes on, so that the fluctuations around the state of maximum entropy should go to zero. Notice also that the hypothesis of equiprobable microstates is too strong for our purpose. In fact the only requirement is that the probability distribution of microstates is sufficiently spread, so that it does not break the property of overwhelming concentration of microstates near the optimal macrostate.

Thus we are lead to solve the following variational problem: find the set of functions $e^* = (e_1^*, \ldots, e_n^*)$ which realizes the maximum value of the functional

$$S(e) = -\int_\Omega \sum_i e_i(\boldsymbol{x}) \log e_i(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x},$$

under the constraints:

$$\sum_i e_i(\boldsymbol{x}) = 1 \quad \text{for all } \boldsymbol{x},$$

$$F_i(e) = \int_\Omega e_i(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = |\Omega_i|, \quad i = 1, \ldots, n,$$

$$E\Big(\sum_i a_i e_i(\boldsymbol{x})\Big) = E(\omega_0),$$

and in the case of a circular domain the supplementary constraint

$$\frac{1}{2}\int_\Omega \sum_i a_i(R^2 - \boldsymbol{x}^2)\, e_i(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \frac{1}{2}\int_\Omega (R^2 - \boldsymbol{x}^2)\, \omega_0(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.$$

This variational problem is studied in §5.

4. Mathematical formulation of the general case

We consider now the general case of an initial vorticity function ω_0 belonging to the space $L^\infty(\Omega)$. Our purpose is to show that the set of all the microstates ξ which have the same constants of the motion as ω_0 is concentrated about a macrostate, and

to determine this macrostate. We need first to give a precise mathematical definition of a macrostate and of this concentration property.

4.1. *Young's measures*

The mathematical definition of a macrostate will be a mapping $\nu: \boldsymbol{x} \to \nu_{\boldsymbol{x}}$ from Ω into the space $M_1(\mathbb{R})$ of all probability measures on $\mathbb{R}$. At each point $\boldsymbol{x}$, we have a probability distribution of vorticity levels. Such a formulation has been used by Young (1942) to solve problems from the calculus of variations, and they are usually called Young's measures. We denote the local average of any continuous bounded function $F(\boldsymbol{x}, a)$ on $\Omega \times \mathbb{R}$ by

$$\langle \nu_{\boldsymbol{x}}, F(\boldsymbol{x}, \cdot) \rangle = \int_{\mathbb{R}} F(\boldsymbol{x}, a)\, \mathrm{d}\nu_{\boldsymbol{x}}(a).$$

4.2. *The neighbourhood of a macrostate*

The rationale which underlies our definition of a macrostate is that the fine-scale oscillations of a microstate $\xi(\boldsymbol{x})$ can be described by a local probability distribution. Then for a bounded continuous test function $F(\boldsymbol{x}, a)$, the integral $F(\boldsymbol{x}, \xi(\boldsymbol{x}))$ on the domain Ω will be approximated by the integral of the local average

$$\int_{\Omega} F(\boldsymbol{x}, \xi(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x} \approx \int_{\Omega} \langle \nu_{\boldsymbol{x}}, F(\boldsymbol{x}, \cdot) \rangle\, \mathrm{d}\boldsymbol{x}.$$

Therefore, we have the following definition:
For $\epsilon > 0$ and any finite set $F_1, \ldots, F_k$ of continuous bounded functions on $\Omega \times \mathbb{R}$, we define the neighbourhood $V_{\epsilon, F_1, \ldots, F_k}(\nu)$ of a macrostate ν as the set of the microstates $\xi(\boldsymbol{x})$ satisfying

$$\left| \int_{\Omega} F_j(\boldsymbol{x}, \xi(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x} - \int_{\Omega} \langle \nu_{\boldsymbol{x}}, F_j(\boldsymbol{x}, \cdot) \rangle\, \mathrm{d}\boldsymbol{x} \right| < \epsilon \quad \text{for all } j = 1, \ldots, k.$$

If we choose a small ϵ and a sufficiently rich set of test functions, the microstates of this neighbourhood will be closely approximated by the macrostate ν, as far as their local mean properties only are considered.

4.3. *Constants of the motion and macrostates*

Each constant of the motion will impose a corresponding constraint on the macrostates that can be approached by the reordering of the initial state. Indeed, if the constant of the motion is of the form $\int_{\Omega} F(\boldsymbol{x}, \xi(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x}$, the macrostate must satisfy

$$\int_{\Omega} \langle \nu_{\boldsymbol{x}}, F(\boldsymbol{x}, \cdot) \rangle\, \mathrm{d}\boldsymbol{x} = \int_{\Omega} F(\boldsymbol{x}, \omega_0(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x}. \tag{2}$$

Otherwise, taking ϵ smaller than the difference between these two quantities, the neighbourhood $V_{\epsilon, F}(\nu)$ would not contain any microstate ξ with the same constant of the motion as the initial state ω_0.

Taking $F(\boldsymbol{x}, \xi(\boldsymbol{x})) = \frac{1}{2}(R^2 - \boldsymbol{x}^2)\, \xi(\boldsymbol{x})$, the relation (2) gives the constraint on the angular momentum of the macrostate. The expression for the energy does not take exactly the same form, but a similar constraint can be written:

$$E(\bar{\nu}) = \frac{1}{2} \int_{\Omega} \bar{\nu}(\boldsymbol{x})\, \Psi(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = E(\omega_0),$$

where $\overline{\nu(x)} = \int_{\mathbb{R}} a \, d\nu_x(a)$ is the local mean vorticity and Ψ the corresponding stream function (defined by $-\nabla^2 \Psi = \bar{\nu}$ with $\Psi = 0$ on the boundary of the domain Ω). In other words, the energy of a macrostate ν is given by the mean vorticity $\bar{\nu}$, without any contribution of the local vorticity fluctuations. The mathematical reason for this property is the complete continuity of the map $\omega \to \boldsymbol{u}$ given by the solution of (1*b*). In the following, when we mention the constraint on that energy, this will also imply the possible constraint on the angular momentum.

The other conserved functionals $I_f(\xi)$ lead to relation (2) with $F(x, \xi(x)) = f(\xi(x))$:

$$\int_\Omega \langle \nu_x, f(\cdot) \rangle \, dx = \int_\Omega f(\omega_0(x)) \, dx.$$

This relation, valid for any continuous function f *on* $\mathbb{R}$, can be written in the more compact form, using the measure π_ω defined in §2:

$$\int_\Omega \nu_x \, dx = \pi_{\omega_0}.$$

We shall say that a macrostate ν which satisfies this relation is a mixing of the vorticity ω_0. So we see how the constants of the motion bring constraints to the macrostates about which the microstates can possibly concentrate; they have to be a mixing of the vorticity ω_0 with the right energy $E(\omega_0)$ (and momentum).

4.4. *The concentration property*

We denote by $\mathscr{X}$ an equipartition of Ω, that is, a finite partition of Ω into subsets Ω_i of equal area. We denote $n(\mathscr{X})$ the number of elements of $\mathscr{X}$ and define the diameter of $\mathscr{X}$:

$$\delta(\mathscr{X}) = \sup_i \delta(\Omega_i), \quad \text{where } \delta(\Omega_i) = \sup_{x, x' \in \Omega_i} |x - x'|.$$

For any equipartition $\mathscr{X}$, we define a random microstate $\omega_{\mathscr{X}}$ by the expression

$$\omega_{\mathscr{X}} = \sum_i A_i \, l_{\Omega_i},$$

where A_i are $n(\mathscr{X})$ equally distributed independent random variables with common distribution $(1/|\Omega|)\, \pi_{\omega_0}$, and l_{Ω_i} denotes the characteristic function of the set Ω_i (equal to 1 inside Ω_i, and 0 elsewhere). A choice of the elementary events is inherent in any probability theory: this choice corresponds to the generalization of the method used in §3, and it will be justified below in the remarks.

We can then give a definition of the concentration property. We shall say that *a set* A *of microstates is concentrated about a macrostate* ν^* *if*: For every neighbourhood $V_{\varepsilon, F_1 \dots F_k}(\nu^*)$, there is a number $\alpha > 0$ such that for any equipartition $\mathscr{X}$ with small enough diameter $\delta(\mathscr{X})$, the conditional probability

$$\text{Prob}\,(\omega_{\mathscr{X}} \notin V_{\varepsilon, F_1 \dots F_k}(\nu^*) \,|\, \omega_{\mathscr{X}} \in A)$$

is less than $\exp(-n(\mathscr{X})\,\alpha)$.

Of course we can replace ν^* by any set $\mathscr{E}^*$ of macrostates in this definition.

Then, our programme is to consider the set A of all the reorderings of ω_0 which have the same energy, that is

$$A = \{\xi \in L^\infty(\Omega) \,|\, \pi_\xi = \pi_{\omega_0} \quad \text{and} \quad E(\xi) = E(\omega_0)\}.$$

The set $\mathscr{E}^*$ will be found by maximizing the Kullback entropy that we define below.

Let us denote by π the *most mixed macrostate*, given by $\pi_x = (1/|\Omega|)\,\pi_{\omega_0}$, for all x. It is of course a mixing of ω_0, but generally has not the same energy as ω_0. The Kullback entropy of ν with respect to π is defined by the formula

$$K_\pi(\nu) = -\int_\Omega \left(\int_{\mathbb{R}} \log \frac{\mathrm{d}\nu_x}{\mathrm{d}\pi_x} \mathrm{d}\nu_x \right) \mathrm{d}x,$$

where $\mathrm{d}\nu_x/\mathrm{d}\pi_x$ is the standard notation for the density of ν_x with respect to π_x (i.e. the ratio of the two probability densities $\nu_x = (\mathrm{d}\nu_x/\mathrm{d}\pi_x)\,\pi_x$), and we use the convention that $\int_{\mathbb{R}} \log(\mathrm{d}\nu_x/\mathrm{d}\pi_x)\,\mathrm{d}\nu_x = +\infty$, if ν_x is not absolutely continuous with respect to π_x (i.e. has no density function). The non-negative functional $\int_{\mathbb{R}} \log(\mathrm{d}\nu_x/\mathrm{d}\pi_x)\,\mathrm{d}\nu_x$ is a classical indication of the 'distance' between the two probability measures ν_x and π_x. Therefore $K_\pi(\nu)$ indicates how far the macrostate ν is from the most mixed state.

The method then relies on the following two results (which are proved by use of the large-deviation theory, see Robert 1989, 1990, 1991):

Concentration theorem. The microstates of the set A are concentrated about the set $\mathscr{E}^*$ of all the macrostates ν^* that realizes the maximum value of $K_\pi(\nu)$:

$$K_\pi(\nu^*) = \max\{K_\pi(\nu) \mid \nu \text{ mixing of } \omega_0 \text{ and } E(\bar{\nu}) = E(\omega_0)\}.$$

Invariance theorem. The concentration property is conserved by the Eulerian flow. That is, if a set Θ of microstates is concentrated about a set $\mathscr{E}$ of macrostates, then, for all t, $\Phi_t(\Theta)$ is concentrated about $\Phi_t(\mathscr{E})$, where, for a macrostate ν, we define $\Phi_t(\nu)$ by

$$\Phi_t(\nu)_x = \nu_{\phi_t^{-1}(x)},$$

ϕ_t being the Lagrangian flow associated with the initial vorticity $\bar{\nu}$. (Heuristically, we may say that the fine-scale oscillations are merely frozen and convected by the mean flow.)

4.5. *Remarks*

(i) Notice that our notion of concentration depends on the choice of a basic probability measure $\pi_0 = (1/|\Omega|)\,\pi_{\omega_0}$ on $\mathbb{R}$. Thus it would be more accurate to say, in the conclusion of the concentration theorem, that the set A is π_0-concentrated about $\mathscr{E}^*$. But, because we work with macrostates which are mixtures of ω_0, to a large extent this concentration properties does not depend on this particular choice of π_0. Indeed, we can see that A is also π_1-concentrated about $\mathscr{E}^*$ for any probability distribution π_1 which satisfies

$$\pi_1 = \rho(a)\,\pi_0, \quad \text{with} \int_{\mathbb{R}} \log \rho(a)\,\mathrm{d}\pi_0(a) > -\infty.$$

Let us denote as $\mathscr{E}$ the set of mixtures of ω_0 such that $E(\bar{\nu}) = E(\omega_0)$, and π' the macrostate $\pi'_x = \pi_1$ for all x. Then, for all ν in $\mathscr{E}$, we have

$$\begin{aligned} K_\pi(\nu) &= -\int_\Omega \left(\int_{\mathbb{R}} \log \left(\frac{\mathrm{d}\nu_x}{\mathrm{d}\pi_x} \frac{\mathrm{d}\pi'_x}{\mathrm{d}\pi_x} \right) \mathrm{d}\nu_x \right) \mathrm{d}x \\ &= K_{\pi'}(\nu) - \int_\Omega \left(\int_{\mathbb{R}} \log \rho(a)\,\mathrm{d}\nu_x(a) \right) \mathrm{d}x, \end{aligned}$$

but as $\int_\Omega \nu_x \mathrm{d}x = \pi_{\omega_0}$, the second term is equal to $|\Omega| \int_{\mathbb{R}} \log \rho(a)\,\mathrm{d}\pi_0(a)$ which is a

constant independent of ν. Therefore $\mathscr{E}^*$ is also the subset of $\mathscr{E}$ on which $K_{\pi'}(\nu)$ reaches its maximum value. Then, the concentration theorem applies again and implies that A is π_1-concentrated about $\mathscr{E}^*$.

(ii) The concentration property depends also on the choice of an equipartition as the support of a random microstate. (This property is not really necessary: the repartition of the subset areas needs only to be sufficiently regular in the partition.) This choice of an equipartition is quite natural because of space homogeneity. It is *a posteriori* justified from the dynamics by the invariance theorem, showing that the concentration property is invariant for the Euler flow. This theorem would not apply if the concentration property were artificially produced by the choice of a very inhomogeneous basic partition. The invariance theorem can be viewed as an ersatz of the Liouville theorem which classically provides a natural invariant measure in phase space. In our case, we can only show the consequence of this theorem that is really relevant for a macroscopic description: the invariance of the concentration property by the flow.

(iii) We can show, by standard compacity arguments (Ellis 1985), that the set $\mathscr{E}^*$ is always non-empty. The case where $\mathscr{E}^*$ is not reduced to a point corresponds to a phase transition situation.

Let us return now to our programme. We can easily verify that the set $\mathscr{E}^*$ is Φ_t invariant (i.e. $\Phi_t(\mathscr{E}^*) = \mathscr{E}^*$). This comes straightforwardly from the definition of $\Phi_t \nu$.

If $\nu \in \mathscr{E}^*$, we have $E(\Phi_t \nu) = E(\overline{\Phi_t \nu}) = E(\Phi_t(\bar{\nu})) = E(\bar{\nu}) = E(\nu)$ and

$$\int_\Omega (\Phi_t \nu)_x \, \mathrm{d}x = \int_\Omega \nu_{\phi_t^{-1}(x)} \, \mathrm{d}x = \int_\Omega \nu_{x'} \, \mathrm{d}x' = \pi_{\omega_0},$$

by the change of variable $x' = \phi_t^{-1} x$. So, $\Phi_t \nu$ satisfies the same constraints as ν.

Making the same change of variable $x' = \phi_t^{-1} x$ and noticing that π_x is constant, we get

$$K_\pi(\Phi_t \nu) = K_\pi(\nu).$$

Our assertion follows:

We call $\mathscr{E}_m^* = \{\overline{\nu^*} \mid \nu^* \in \mathscr{E}^*\}$ the equilibrium set. If we suppose that $\mathscr{E}^* = \{\nu^*\}$, we have $\Phi_t \nu^* = \nu^*$ for all t, and thus $\omega^* = \bar{\nu}^*$ is a stationary solution of the Euler system.

5. Computation of the equilibrium states, the equation of Gibbs states

5.1. *Kullback and Boltzmann entropy*

We have now to solve the variational problem: Find the macrostates ν^* satisfying

$$K_\pi(\nu^*) = \max\left\{K_\pi(\nu) \mid \int_\Omega \nu_x \, \mathrm{d}x = \pi_{\omega_0}, E(\nu) = E(\omega_0)\right\}, \tag{3}$$

where ω_0 is any initial vorticity in $L^\infty(\Omega)$. We have seen in the previous remarks that such ν^* always exist.

We begin with the simpler case where ω_0 takes only a finite number of distinct values $a_1, \ldots, a_n$. Then, we have seen that $\pi_{\omega_0} = |\Omega_1| \, \delta_{a_1} + \ldots + |\Omega_n| \, \delta_{a_n}$. It is clear that any mixture of π_{ω_0} must be of the form $\nu_x = e_1(x) \, \delta_{a_1} + \ldots + e_n(x) \, \delta_{a_n}$, with the constraints

$$F_i(e_1 \ldots e_n) = \int_\Omega e_i(x) \, \mathrm{d}x = |\Omega_i|, \quad i = 1 \ldots n.$$

The most mixed state is such that $e_i(x) = |\Omega_i|/|\Omega|$, for every x; that is $\pi_x = (1/|\Omega|)\pi_{\omega_0}$. The probability distribution ν_x is obtained by multiplying π_x by a function of a equal to $e_i(x)|\Omega|/|\Omega_i|$ for $a = a_i$. This function is equal to $d\nu_x/d\pi_x$. Therefore the Kullback entropy is written

$$K_\pi(\nu) = -\int_\Omega \sum_i e_i(x) \log\left(\frac{|\Omega|}{|\Omega_i|} e_i(x)\right) dx$$

$$= -\int_\Omega \sum_i e_i(x) \log e_i(x)\, dx - \sum_i |\Omega_i| \log \frac{|\Omega|}{|\Omega_i|}.$$

Since the second term is a constant, this entropy is indeed equivalent to the Boltzmann entropy, and we find again the variational problem stated at the end of §3.

5.2. *The case of a piecewise-uniform initial vorticity*

As $e = (e_1, \ldots, e_n)$ must satisfy the supplementary constraint $\sum_{i=1}^n e_i(x) = 1$, there remain only $n-1$ independent constraints $F_1, \ldots, F_{n-1}$. Let $e_1^*, \ldots, e_n^*$ be a solution of our variational problem. Then, from the Lagrange multipliers rule, there are constants $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ and β such that the first variations of the functionals satisfy

$$\delta S = \beta \delta E + \sum_{i=1}^{n-1} \alpha_i \delta F_i$$

for all variations δe_i such that $\sum_i \delta e_i = 0$. Straightforward computations give

$$\delta S = -\int_\Omega \sum_i (1 + \log e_i^*)\, \delta e_i\, dx,$$

$$\delta E = \int_\Omega \psi^* (\sum_i a_i \delta e_i)\, dx,$$

where Ψ^* is the stream function associated to the vorticity $\sum_{i=1}^n a_i(e_i^*(x)$, and so

$$\delta F_i = \int_\Omega \delta e_i\, dx.$$

Then, we easily get

$$e_i^*(x) = \frac{\exp(-\alpha_i - \beta a_i \Psi^*(x))}{Z(\Psi^*(x))}, \tag{4}$$

where the partition function Z is given by

$$Z(\Psi) = \sum_{i=1}^n \exp(-\alpha_i - \beta a_i \Psi)$$

and we take $\alpha_n = 0$ by convention.

Therefore Ψ^* satisfies the equation of Gibbs states:

$$\left.\begin{aligned} -\nabla^2 \Psi^* = \sum_i a_i e_i^*(x) = -\frac{1}{\beta} \frac{d}{d\Psi} \log Z(\Psi^*), \\ \Psi^* = 0 \quad \text{on} \quad \partial\Omega. \end{aligned}\right\} \tag{5}$$

So, we see that if e^* is a solution of the variational problem, there are constants α, β such that e_i^* is given by (4) and Ψ^* by (5).

Conversely, for any given set of parameters α, β we can consider a solution $\Psi^{\alpha,\beta}$ of the nonlinear equation (5) (as the right-hand side of the equation is of the form $f(\Psi^*)$, with f continuous and bounded, we know, using Schauder's fixed-point theorem, that a solution, in general not unique, always exists) and the associated Gibbs state $e^{\alpha,\beta}$ given by (4). Of course $e^{\alpha,\beta}$ is a critical point of the functional:

$$J(e) = S(e) - \sum_{i=1}^{n-1} \alpha_i F_i(e) - \beta E(e) \quad \text{on the linear manifold } \sum_i e_i(x) = 1.$$

Furthermore we prove the following result

THEOREM. *If $\beta > -\lambda_1/(\sup_i a_i^2)$, where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\nabla^2$ (associated with the Dirichlet boundary value condition), then $e^{\alpha,\beta}$ is the unique maximum of the functional $J(e)$ on the set defined by $e_i(x) \geqslant 0$, $i = 1 \dots n$, $\sum_i e_i(x) = 1$.*

Proof. We shall prove that the functional $J(e)$ is strictly concave on the set defined by $0 < e_i(x) < 1$, $i = 1 \dots n$. With that aim, let us compute the second variation $\delta^2 J$ for any variation δe_i. Straightforward computations give

$$\delta^2 S = -\frac{1}{2} \int_\Omega \sum_{i=1}^n \frac{(\delta e_i)^2}{e_i} \mathrm{d}\boldsymbol{x},$$

$$\delta^2 E = \frac{1}{2} \int_\Omega \delta\Psi \, \delta\omega \, \mathrm{d}\boldsymbol{x},$$

where we denote $\delta\omega = \sum_i a_i \delta e_i$, and $\delta\Psi$ is the stream function associated to $\delta\omega$. As $\delta^2 F_i = 0$, we obtain

$$\delta^2 J = -\frac{1}{2} \int_\Omega \sum_{i=1}^n \frac{(\delta e_i)^2}{e_i} \mathrm{d}\boldsymbol{x} - \frac{\beta}{2} \int_\Omega \delta\Psi \, \delta\omega \, \mathrm{d}\boldsymbol{x}.$$

Let us consider first the case $\beta \geqslant 0$. Then we have, by Green's formula,

$$\int_\Omega \delta\Psi \, \delta\omega \, \mathrm{d}\boldsymbol{x} = \int_\Omega (\nabla \delta\Psi)^2 \, \mathrm{d}\boldsymbol{x} \geqslant 0,$$

from which

$$\delta^2 J \leqslant -\frac{1}{2} \int_\Omega \sum_{i=1}^n \frac{(\delta e_i)^2}{e_i} \mathrm{d}\boldsymbol{x} < 0;$$

thus J is strictly concave.

We consider now the case $\beta < 0$. Using the well-known inequality

$$\int_\Omega \delta\Psi \, \delta\omega \, \mathrm{d}\boldsymbol{x} \leqslant \frac{1}{\lambda_1} \int_\Omega (\delta\omega)^2 \, \mathrm{d}\boldsymbol{x}$$

we get

$$\delta^2 J \leqslant -\frac{1}{2} \int_\Omega \left[\sum_i \frac{(\delta e_i)^2}{e_i} + \frac{\beta}{\lambda_1} \left(\sum_i a_i \delta e_i \right)^2 \right] \mathrm{d}\boldsymbol{x}.$$

We have

$$\left(\sum_i a_i \delta e_i \right)^2 \leqslant \left(\sup_i a_i^2 \right) \left(\sum_i |\delta e_i| \right)^2,$$

and Cauchy–Schwartz inequality gives

$$\left(\sum_i |\delta e_i| \right)^2 = \left(\sum_i \frac{|\delta e_i|}{e_i^{\frac{1}{2}}} e_i^{\frac{1}{2}} \right)^2 \leqslant \sum_i \frac{|\delta e_i|^2}{e_i} \sum_i e_i = \sum_i \frac{(\delta e_i)^2}{e_i},$$

from which $$\delta^2 J \leqslant -\tfrac{1}{2}\left[1+\frac{\beta}{\lambda_1}(\sup_i a_i^2)\right]\int_\Omega \sum_i (\delta e_i)^2 \,\mathrm{d}\boldsymbol{x}.$$

And we see that for $\beta > -(\lambda_1/\sup_i a_i^2)$, $\delta^2 J$ is < 0. As $e^{\alpha,\beta}$ is a critical point of J, it is the unique point on which J reaches its maximum value.

Remarks. (i) A consequence of this result is that, for $\beta > -\lambda_1/(\sup_i a_i^2)$, the Gibbs state $e^{\alpha,\beta}$ is a maximum of $S(e)$ subject to the constraints

$$\sum e_i = 1, \quad E(e) = E(e^{\alpha,\beta}), \quad F_i(e) = F_i(e^{\alpha,\beta}), \quad i = 1 \ldots n.$$

(ii) As usual, we replace the consideration of the constrained maximization problem for $S(e)$ by the consideration of the equation of Gibbs states, equation (5). The general consideration of (5) for all values of the parameters α, β is far from obvious, because many bifurcation phenomena can occur (see Sommeria *et al.* 1991).

(iii) As β is the Lagrange multiplier of the energy constraint, it is the inverse of a temperature. And the above theorem proves the existence of equilibrium states with a negative temperature (this phenomenon was foreseen by Onsager (1949)).

(iv) When some supplementary constants of the motion occur, we must take them into account. This leads, of course, to some modifications in the equation of Gibbs states. For example, when Ω is the ball $B(0,R)$ we must consider the angular momentum

$$M = \frac{1}{2}\int_\Omega (R^2 - \boldsymbol{x}^2)\,\omega(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Then the Lagrange multipliers rule introduces a new multiplier γ:

$$\delta S = \beta \delta E + \sum \alpha_i \,\delta F_i + \gamma \delta M,$$

and we get the new Gibbs states

$$-\nabla^2 \Psi = \frac{\sum a_i \exp\{-\alpha_i - a_i[\beta\Psi + \frac{1}{2}\gamma(R^2 - \boldsymbol{x}^2)]\}}{Z}, \tag{6}$$

$$\Psi = 0 \quad \text{on} \quad \partial\Omega,$$

where $Z = \sum \exp\{-\alpha_i - a_i[\beta\Psi + \frac{1}{2}\gamma(R^2 - \boldsymbol{x}^2)]\}$.

Radial solutions $\Psi(r)$ of (6) are stationary solutions of Euler equations, but non-radial solutions, which may occur by a breaking of rotation invariance after a bifurcation (when $-\beta$ becomes large), are generally not stationary, owing to the presence of the term in $R^2 - \boldsymbol{x}^2$. The change of stream function $\phi = \Psi + (\gamma/2\beta)(R^2 - \boldsymbol{x}^2)$, which corresponds to a change of reference frame (the new frame being in uniform rotation), gives again stationary solutions.

5.3. *The general case*

The general case of any measurable and bounded initial vorticity ω_0 can be studied in a similar way. In yields the generalization of (4) for the optimum state ν (we omit the star that labels the optimum state):

$$d\nu_x(a) = \frac{\exp[-\alpha(a) - \beta a\,\Psi(\boldsymbol{x})]}{Z(\Psi(\boldsymbol{x}))}\,\mathrm{d}\pi_0(a),$$

where $Z(\Psi) = \int \exp[-\alpha(a) - \beta a\,\Psi]\,\mathrm{d}\pi_0(a)$, and $\pi_0 = (1/|\Omega|)\,\pi_{\omega_0}$. The Lagrange multipliers $\alpha_1, \ldots, \alpha_n$ are now replaced by the continuous function $\alpha(a)$. Notice that

$(\mathrm{d}\nu_x/\mathrm{d}a)(a)$ is the probability density for finding the vorticity a at position $\boldsymbol{x}$, and $\mathrm{d}\pi_0/\mathrm{d}a$ the global density of the initial vorticity distribution.

Then the generalized Gibbs state can be written:

$$-\nabla^2\Psi = -\frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}\Psi}\log Z, \tag{7}$$

$$\Psi = 0 \quad \text{on} \quad \partial\Omega,$$

with Ψ also satisfying the following constraints: global conservation of vorticity

$$\int_\Omega \frac{\exp[-\alpha(a)-\beta a\,\Psi(x)]}{Z(\Psi(x))}\mathrm{d}\boldsymbol{x} = |\Omega| \quad \text{for all } a;$$

conservation of energy $$\frac{1}{2}\int_\Omega \Psi\nabla^2\Psi\,\mathrm{d}\boldsymbol{x} = -E(\omega_0).$$

General properties of this generalized Gibbs state can be obtained. Let us denote

$$z(\chi) = \int_{\mathbb{R}} \exp[-\alpha(a)+\chi a]\,\mathrm{d}\pi_0(a).$$

Then the right-hand side of (7) can be written

$$f(\Psi) = \frac{\mathrm{d}}{\mathrm{d}\chi}\log z(-\beta\Psi).$$

The function $\log z(\chi)$ is strictly convex; indeed, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}\chi^2}\log z = \frac{1}{z^2}\left[z\frac{\mathrm{d}^2 z}{\mathrm{d}\chi^2} - \left(\frac{\mathrm{d}z}{\mathrm{d}\chi}\right)^2\right],$$

and this last term is strictly positive by application of the Cauchy–Schwarz inequality:

$$\left(\int a\exp\left[\frac{-\alpha(a)+\chi a}{2}\right]\exp\left[\frac{-\alpha(a)+\chi a}{2}\right]\mathrm{d}\pi_0\right)^2$$
$$< \left(\int \exp[-\alpha(a)+\chi a]\,\mathrm{d}\pi_0\right)\left(\int a^2\exp[-\alpha(a)+\chi a]\,\mathrm{d}\pi_0\right).$$

Then we get the striking result that for a Gibbs state, the function $\bar{\omega} = f(\Psi)$ is either strictly decreasing (case $\beta > 0$), or strictly increasing (case $\beta < 0$), or perhaps a constant (if $\beta = 0$). Furthermore, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}\chi^2}\log z = \frac{z''}{z} - \left(\frac{z'}{z}\right)^2 \leqslant \frac{z''}{z} \leqslant \max\{a^2/a \in \operatorname{supp}\pi_0\},$$

and we can prove the following (Robert 1991):

PROPOSITION. *For* $-\beta\max\{a^2 \mid a \in \operatorname{supp}\pi_0\} < \lambda_1$, *there is a unique solution to* (7).

5.4. *Isolated vortex structures*

Fairly isolated vortex structures are often observed in two-dimensional turbulent flows. As already foreseen by Onsager (1949), such vortices must correspond to a negative temperature. Indeed, a vortex core corresponds to an extremum of the

stream function, $\nabla^2\Psi$ is positive for a minimum and negative for a maximum. Let us assume that Ψ is maximum, and therefore the vorticity $-\nabla^2\Psi$ is positive. Assume also that Ψ vanishes on the walls, then Ψ is positive. Now, if β is positive, $\bar{\omega} = f(\Psi)$ must be a decreasing function of Ψ so that $\bar{\omega}$ would increase from the vortex core to the wall. The vorticity would be essentially located at the walls. For β negative the vorticity is concentrated in the vortex core, as expected. When Ψ is minimum in the vortex core, we only have to reverse the sign to get the same conclusion.

Let us consider the special case of a structure produced by a localized initial vorticity excitation, which reaches a stable configuration after strong mixing with the surrounding irrotational fluid. In this situation, which we call the dilute case, the main contribution to the vorticity distribution π_0 comes from the vorticity level $a = 0$, so that π_0 is close to the Dirac distribution centred in 0, and $Z(\Psi) \approx \exp[-\alpha(0)] = Z(0)$; then we get the approximate Gibbs equation

$$-\nabla^2\Psi = \frac{1}{Z(0)}\frac{\mathrm{d}z}{\mathrm{d}\chi}(-\beta\Psi) = f(\Psi).$$

Since we are interested in localized vortices, we consider only the case $\beta < 0$. We see immediately that all the even derivations of $z(\chi)$ are positive. Therefore the vorticity is an increasing function of Ψ, whose second derivative is also increasing.

When the initial state is dominated by the positive vorticity, all the derivatives of $z(\chi)$ become positive, so $f(\Psi)$ is a convex function very similar to an exponential. When the contribution of positive and negative vorticity is symmetric, we may assume that $\alpha(-a) = \alpha(a)$, and we can write

$$z(\chi) = 2\int_0^\infty \exp[-\alpha(a)]\cosh(\chi a)\,\mathrm{d}\pi_0(a).$$

Therefore, the second derivative of f has the same sign as Ψ: the function f is convex for positive Ψ and concave for negative Ψ, it behaves very much like a sinh function. Of course, these results apply only for the dilute case. In general the effect of the denominator $Z(\Psi)$ in equation (7) is to make the vorticity saturate in the vortex cores, so that it never exceeds the initial extrema.

6. The role of viscosity and the stability of the equilibrium states

We can take into account the effect of a very small viscosity by assuming that first the flow develops inertial vorticity structures at very fine scale, and then the supplementary term $-(1/Re)\nabla^2\omega$ has a filtering effect on the small-scale oscillations of the vorticity $\omega(t, x)$. Thus the final flow can be assimilated with the local average $\bar{\omega}$ of our theory. It was shown in §3 or §4.3 that the energy of this final mean flow must be equal to the energy of the initial field ω_0. Furthermore, the equilibrium macrostate is a mixing of the initial state, and all the functions of the vorticity are globally conserved in this inertial phase. The enstrophy is thus globally conserved but it goes partly into the local fluctuations and partly into the mean vorticity $\bar{\omega}$. Thus, denoting as $\overline{\omega^2}$ the local average of the squared vorticity,

$$\int_\Omega \bar{\omega}^2\,\mathrm{d}x \leqslant \int_\Omega \overline{\omega^2}\,\mathrm{d}x = \int_\Omega \omega_0^2\,\mathrm{d}x.$$

If the fluctuations are suppressed by a small viscosity, their enstrophy is dissipated, and only the contribution of the mean flow $\bar{\omega}$ remains. The same inequality applies

for higher even momenta of vorticity. Therefore our theory is consistent with the usual distinction between the rugged and dissipated integrals: the energy, total circulation and momentum are still conserved in the presence of a small viscosity, while enstrophy and higher-order vorticity momenta are dissipated. Notice that this irreversibility is already present in the purely inviscid theory: part of the enstrophy of the initial ordered state goes to the microstate fluctuations. This is analogous to the irreversible transfer of mechanical energy into thermal motion which appears in ordinary statistical mechanics.

Then we may wonder whether the equilibrium state has some stability property. What do we get as equilibrium state if we repeat the process and take $\omega^* = \bar{\omega}$ as initial vorticity? Let Ψ^* be the unique solution of the general Gibbs state equation, with ω^* the corresponding vorticity. Then, it can be shown that Arnol'd's classical estimates (Arnol'd 1969) apply and give the following stability result: any bounded initial vorticity ω_0 gives a solution $\omega(t)$ of the Euler equations, which satisfies the inequality

$$\int_\Omega (\omega(t)-\omega^*)^2 \,\mathrm{d}x \leqslant C \int_\Omega (\omega_0-\omega^*)^2 \,\mathrm{d}x \quad \text{for all } t,$$

where C is some > 0 constant.

Furthermore, we can prove that if ν is a mixture of ω^* such that $E(\nu) = E(\omega^*)$, then, for all x, $\nu_x = \delta_{\omega^*(x)}$. So, if we repeat the process, starting with ω^* we shall get ω^* again as the equilibrium state.

7. Discussion

This theory provides clear formulae that can be compared to numerical simulations or experiments. In a strictly inviscid fluid, the theory predicts the local statistical distribution of vorticity levels at any given time after a long evolution, beyond any cascade processes. A scale separation between the mean flow and the vorticity fluctuations is implicit in our macroscopic description. This is why the closure problem can be solved. This choice for the macroscopic description is justified by the concentration theorem. There are more and more possible microscopic states as we consider fields with smaller and smaller scales. Therefore it is not surprising that the microstates concentrate toward a macrostate.

When the vorticity fluctuations are smoothed out by a weak viscosity, a relationship between vorticity and stream function in the final steady state is predicted. However, the equations which define this equilibrium state have generally no explicit solutions, even in the restricted case of an initial condition with a few uniform vorticity patches (except for a case considered by Sommeria *et al.* 1991). A numerical exploration for simple cases is in progress, but this is made difficult by the presence of bifurcations and multiple solutions. The Lagrange multipliers have to be related to the constants of the motion, which are set by the initial condition of the flow.

Another possible test is to measure directly the curve $\omega = f(\psi)$ in experiments or numerical simulations and compare with theory, considering the Lagrange parameters α_i and β as fitting parameters. By contrast with earlier theories of optimal vortex structures, like the minimum-enstrophy vortices of Leith (1984), the present theory predicts a final state and relationship f that do depend on the initial values of all the constants of the motion. A direct quantitative comparison with theory is possible only for initial conditions with a very limited number of initial

vorticity levels. Otherwise the number of adjustable parameters (α_i and β) would be too big for a significative test. Because of these constraints, we could not find any convenient example in the literature where such quantitative tests could be done. Therefore we have performed numerical simulations of a shear layer starting as a band of uniform vorticity. The results, presented in the paper by Sommeria *et al.* (1991), are found to show excellent agreement with theory. Laboratory experiments are also in progress, in which the two-dimensional flow is produced in a layer of mercury submitted to a transverse magnetic field. The flow is generated in a circular box by an appropriate distribution of electric current, and the final steady structure resulting from a complex flow evolution is studied. A good fit of $f(\psi)$ with the predicted relationship is also obtained. However, with some initial conditions new vorticity levels are brought into the domain by boundary-layer detachment. In that case, the relation with the theory for Euler flows is not clear, even at large Reynolds number.

In many interesting instances of spontaneous flow organization, isolated vortex structures are produced, far from the boundaries of the fluid domain. These monopoles, dipoles or even tripoles are observed in numerical computations (Legras, Santangelo & Benzi 1988) or laboratory experiments (van Heijst & Kloosterziel 1989). Although the theory presented here is for the case of a bounded domain, it can be extended to the unbounded case, where some supplementary technicalities are needed. In general, such vortex structures arise in a part of a turbulent flow and it is not possible to clearly identify the initial condition. In laboratory experiments, the initial flow is often three-dimensional and becomes two-dimensional under the influence of an external force like stratification or electromagnetic forces. Therefore we expect from the present theory that the final vortex structure is not easily predictable. A precise prediction would require a good control of the initial condition. However, general properties can be deduced, as shown in §5. First of all it was proved that isolated vortex structures must correspond to negative temperature, so that ω is a strictly increasing function of ψ. Most often, such structures result from a strong localized vorticity perturbation, or remain as 'islands' in a background of turbulence with strong mixing and low vorticity levels. Therefore the dilute approximation defined in §5.4 can be applied to these structures. Thus the theory predicts that for monopoles or symmetric dipoles, f' is extremal at the vortex cores (see §5.4). These properties have been noticed by Nguyen Duc & Sommeria (1988) in experiments with vortex pairs, and had remained unexplained. Similar behaviour is also observed in numerical simulations mentioned in that paper. This contrasts with the structure of vortices produced by more gentle processes (Deem & Zabusky 1978), which reach a stable structure close to patches of uniform vorticity. These correspond to cases with little mixing, for which the vorticity saturates at the initial values.

One of the important applications of two-dimensional turbulence is geophysical fluid dynamics, and this theory should provide predictions on the equilibrium flow structures in the atmosphere or ocean. The theory can be readily applied to a spherical geometry, with the advantage that the flow can be bounded without the presence of lateral walls with possible boundary-layer detachment. The presence of a Coriolis force can be easily taken into account in the frame of the quasi-geostrophic approximation. The resulting equations are the two-dimensional Euler equations where the vorticity has been replaced by the potential vorticity. The present theory can be applied to this case as well, by replacing the vorticity by the potential vorticity in equation (5) but keeping the same relation (1*b*) between vorticity and stream function. The extension to a multi-layer system is straightforward. The

theory applies also to more general approximations of slow motion close to geostrophic balance. The models need to conserve a potential vorticity and energy, and to possess an invertibility principle: the whole evolution is determined by the potential vorticity field, and the stream function can be calculated from the vorticity field by inverting a differential operator, like the Laplacian in the quasi-geostrophic approximation. In particular this theory can be applied to the semi-geostrophic model of Hoskins, McIntyre & Robertson (1975), or the more general models proposed by Salmon (1985). It cannot be applied to models that break the invariants of the motion.

Earlier theories of equilibrium statistical mechanics had some success in predicting a variety of atmospheric or oceanic flow regimes (Holloway 1986). Our theory can be applied to these situations, but also accounts for more localized eddies in a unified way. It can be considered as a generalization of the idea of potential vorticity mixing, which has been successful in explaining the general oceanic circulation (Rhines 1986). Our theory would reduce to the mixing of potential vorticity in the absence of constraints due to energy conservation.

In most cases of geophysical interest, the flow structures do not arise in the final stage of a flow evolution, but are permanently forced. However, we can consider the case of *inertial structures*, which are controlled by advective effects, while a weak forcing maintains the flow against dissipation. This is surely an excellent approximation in many cases, such as the oceanic currents, the atmospheres of giant planets, or the Earth's stratosphere. Friction and forcing effects are more important in the Earth weather systems, but a description in terms of potential vorticity transport is still a good approach, as discussed by Hoskins *et al.* (1985). The present theory can be expanded to such weakly forced cases using an adiabatic approximation: the flow stays in statistical equilibrium but the Lagrange multipliers slowly vary with time. This could provide a frame for a modelling of atmospheric regimes and climate.

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